Discrete spacetime: classical causality, prediction, retrodiction and the mathematical arrow of time[*]

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Abstract

A mathematical definition of classical causality over discrete spacetime dynamics is formulated. The approach is background free and permits a definition of causality in a precise way whenever the spacetime dynamics permits. It gives a natural meaning to the concepts of cosmic time, spacelike hypersurfaces and timelike or lightlike flows without assuming the notion of a background metric. The concepts of causal propagators and the speed of causality are discussed. In this approach the concepts of spacetime and dynamics are linked in an essential and inseparable whole, with no meaning to either on its own.

1   Introduction

The term *causality* is frequently used in a way which suggests that an intrinsic causal structure underlies the universe. In relativity this is reinforced by the assumption of a metric tensor with a Lorentzian signature. This gives the traditional light cone structure associated with spacelike and timelike intervals, and imposes conditions on the possible trajectories of particles and quantum field theory operator commutation relations.

We shall discuss the idea that *causality is a convenient account designed to satisfy and conform to the patterns of classical logic that the human Theorist wishes to believe underlies the dynamics of space, time and matter.* In this approach causality need not be associated with any *a priori* concept of metric tensor.

This view of causality has been suggested by various philosophers and scientists. Hume argued that causality is a fiction of the mind. He said that causal reasoning is due to the mind's expectation that the future is like the past and because people have always associated an effect with a cause. Kant believed that causality is a category used to classify experience. Lentzen said that causality is a relation within the realm of conceptual objects. Lurchin said that causality is a personal way of thought that is not among our immediate sensual data, but is rather our basic way to organize that data. For Maxwell the principle of causality expresses the general objective of theoretical sciences to achieve deterministic explanations, and according to Heisenberg, causality only applies to mathematical representations of reality and not to individual mechanical systems.

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2 The PPM view of time

To avoid confusion we distinguish three sorts of time:

i) Process Time is the hypothesized time of physical reality. Although there is geological and astrophysical evidence for some sort of temporal ordering in reality [1], process time need not exist in any real sense and may just be a convenient way for humans to think about the Universe.

ii) Physiotime is the subjective time that humans sense and which they believe runs monotonically forwards for them. It is the end product of complex bio-dynamical processes occurring in process time. Its origins are not understood currently. Many physicists believe that this feeling is an illusion. What matters here is the undeniable *existence* of this feeling, because humans are driven by this sensation of an ever increasing time to believe that descriptions of reality must involve such a concept.

iii) Mathematical times are conceptual inventions of human theorists designed to model process time. Examples are Newtonian Absolute Time, relativistic coordinate times, proper time and cosmic time. Mathematical times usually involve subsets of the real line, which has an ordering property. This ordering is used to model the notions of *earlier* and *later.* This presupposes something about the nature of process time that may be unwarranted. In the Euclidean formulation of field theory for example there is no dynamical ordering parameter.

2.1 The Theorist

We shall use the term *Theorist* to denote the human mind operating at its clearest and most rational in physiotime. The Theorist has the status of an observer or deity overseeing the mathematically consistent development of chosen mathematical models that are used to represent phenomena in process time. *Free will* enters into the discussion here as the freedom of the Theorist to choose boundary conditions in these models. Whether free will is an illusion or not is regarded here as irrelevant.

3 Classical causality

The need to seek causal explanations stems from the peculiarities of human consciousness. Humans generally want to *explain* phenomena. When they do this they invariably try to invoke what may be called *classical logic.* This is the everyday logic that postulates that statements are *either true or else false* and that conclusions can be drawn from given premises. It is also the logic of vision, which generally informs the brain that an object either is in a place or else is not in that place. The rational conscious mind tends to believe that the external universe follows this logic, and this is the basis for the construction of CM (classical mechanics) and all the belief structures which it encodes into its view of reality. It is also the logic of jurisprudence and common sense. This logic served humanity extremely well for millennia, until technological advances in the early years of the twentieth century revealed that quantum phenomena did not obey this logic in detail.

A CM Theorist is anyone who believes in a classical view of reality. In the mindset of a CM Theorist, reality is assumed to be strictly single valued at each and every time even in the absence of observation. Philosophers say that reality is determinate. The CM Theorist attempts to make unique predictions wherever possible, such as where a planet will be at a

future time. The assumption is made that the planet will be somewhere at that time and not nowhere, and that it will not be in two or more places at that time.

In general, quantum theory requires a pre-existing classical conceptual framework for a sensible interpretation. For example, relativistic quantum field theory assumes a classical Lorentzian metric over spacetime, and only the fields are quantised. For this reason, we shall focus our attention on a classical formulation of causality.

4   Functions and links

To set up our framework for causality, it will be useful to review the definition of a *function*:

Definition 1:   A *function* f [2] is an ordered triple $f \equiv (F, D, R)$ where F, D and R are sets which satisfy the following:

1. F is a subset of the Cartesian product $D \times R$;

2. for each element x in D there is exactly one element y in R such that the ordered pair (x, y) is an element of F.

Physicists tend to write $y = f(x)$. D is called the *domain* of (definition of) f and R is the *range* of f. The *image* of f is the subset $f(D)$ of R such that for each element v in $f(D)$ there is at least one element u in D such that $v = f(u)$.

Without further information it cannot be assumed that $f(D) = R$, but in our work this must be assumed to hold. Otherwise, there arises the possibility of having a CM where something could happen without a cause, i.e. an element z of R could exist for which there is no x in D such that $z = f(x)$.

The range and domain of a function do not have a symmetrical relationship and the ordering of the sets in *Definition* 1 is crucial. Usually, no pair of component sets in the definition of a function can be interchanged without changing the function. This asymmetry forms the basis of the time concept discussed in this article and defines what we call a *mathematical arrow of time*. Assuming that f is single valued, then we may employ the language of dynamics here, though this may seem unusual. We could say that $y = f(x)$ is *determined* by x *via the process* f, or that x *causes* $f(x)$. Then x is a *cause*, $f(x)$ is its *effect* and f is the *mechanism of causation*.

Although *Definition* 1 carefully excludes the concept of a many-valued explicit function, such a possibility arises when we discuss implicit functions.

Graphical notation: We shall use the following graphical notation:

4.1   Explicit functions:

The process of mapping elements of D into R via f will be represented by the LHS (left hand side) of $Fig.1a$, where the large circles denote domain and image sets, the small circle denotes the function, and arrows indicate the direction of the mapping. An alternative representation is given by the RHS (right hand side) of $Fig.1a$, where the labels D, R and f are understood. Here the integers 0 and 1 represent the ordering of the function *from D* $(\equiv 0)$ *to* $R (\equiv 1)$, so that the arrows are not needed.
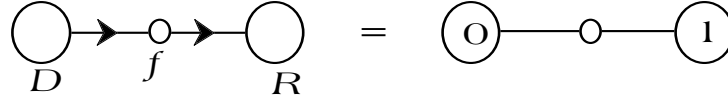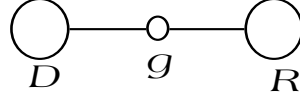
Figure 1: (a) The LHS represents a function f with domain D and range R; the RHS is an equivalent representation of the same with D, R and f understood. The ordering of the digits 0, 1 represents the direction of the mapping. (b) A representation of an implicit function g of two variables.

4.2 Many-to-one functions:

Definition 1 raises the question: *which particular x in D caused a given y in R?* As given, nothing in *Definition* 1 rules out many-to-one mappings, so without further information about the function f, there may be more than one such x. The assumption of a unique pre-image is equivalent to the belief that, given the present state of the universe, there was a unique past which gave rise to it. 't Hooft has recently discussed this in the context of gravitation using equivalence classes of causes [3]. This is bound up with the notion of irreversibility.

If however it is the case that f is one-to-one and onto, then its inverse function f^{-1} exists and so any element $y = f(x)$ in R can be mapped back to a unique cause x in D via f^{-1}. In such a case there would be no inherent difference in principle between the roles of D and R. In the language of dynamics the mechanics would be *reversible.*

4.3 Implicit functions and links:

Suppose now that the relationship between D and R is implicit rather than explicit. For example, let D and R each be a copy of the real line R with elements x, y belonging to D and R respectively and suppose that the only dynamical information given was an implicit equation of the form

$$g(x, y) = 0. \tag{1}$$

Then our graphical notation for this is given by $Fig.1b$, where the small circle now denotes an implicit equation or *link* g relating elements of D and R. Without further information no arrows are permitted at this stage.

4.4   Resolution of links:

Given an implicit equation in two variables such as (1), suppose now that it could proved that there was always a unique solution for y in R given any x in D (the solution y of course depending on the value of x). Then our convention is that now arrows pointing from D into R via the link g may be added to indicate this possibility, giving the LHS of $Fig.1a$ with f replaced by g. But now the relationship between D and R is formally equivalent in principle to having an explicit function y of x (even if y could not be obtained analytically). In such a case we will also say that any given element of D *causes* or *determines* a corresponding element of R and that the link between D and R may be *resolved* from D *to* (or in favour of) R.

Our concept of resolution depends only on the *existence* of a unique solution and does not imply that a solution could actually be computed by the Theorist in practice. Computability is an attribute associated with *physiotime*, and is not here regarded as an essential ingredient of our version of causality. A more severe definition of causality however might be to impose the restriction of computability. We do not do this here because we wish to avoid anthropomorphism. What is important in classical mechanics is the existence of a unique resolution; the universe does not actually "compute" anything when this resolution occurs.

4.5   Inequalities:

It is possible to consider links which are not equations but more general relations such as inequalities. For example, suppose D and R are copies of the real line with elements denoted by x, y respectively and consider a link defined by

$$g(x,y) \equiv x + y < 0. \tag{2}$$

Given x there is an infinity of solutions for y, so in this case one possible interpretation would be that the link is equivalent to a many valued function of x, although this way of putting things might seem unusual. On the other hand, given a y there is also an infinity of solutions for x. This leads to an alternative interpretation of the link as a many valued function of y.

Such examples do not generate a classical TRA (temporal resolution of alternatives) and so would not occur normally in our classical spacetime dynamics.

4.6   Reversibility:

Now suppose we were given an implicit equation for x and y such that we could prove the existence of a unique solution for either x or y given the other variable. Then the arrows could point either way and this would correspond to a choice of causation[1]. Because cause and effect can be interchanged in such a case, it then becomes meaningful to talk about this dynamics being *reversible.* Clearly this is formally equivalent to having an invertible explicit function.

When discussed in this way it becomes clear that in general, spacetime dynamics will be irreversible. Reversibility will occur only under very special conditions, which of course is the experience of experimentalists.

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[1]This choice is taken by the Theorist in physiotime.

4.7   Generalization:

In general the sets D and R need not be restricted to the reals. They could be any sort of sets, such as vector spaces, tensor product spaces, quaternions, operator rings, group manifolds, etc. Whatever their nature, they will always be referred to as *events* for convenience. In the sorts of dynamics we have in mind events will often be sets such as group manifolds.

Links are defined as specific relationships between events. They may be more complicated than those in the above examples, and may have several or many components relating different components of different events. The specification of a set of events and corresponding links will be said to specify a (classical) *spacetime dynamics.*

An important generalization is that a link might involve more than two events. Given for example five events R, S, T, U and V with elements r, s, t, u and v respectively then a classical dynamics involving these events would give some set of equations or link g of the generic form

$$g\left(r, s, t, u, v\right) = 0. \tag{3}$$

This will be represented by *Fig.2a*.

Suppose now, given such an g, that it could be proven that there is always a unique solution $t \in T$ given the other elements r, s, u and v. In such a case this will be indicated by arrows pointing from R, S, U and V into g and an arrow pointing into T as in the LHS of *Fig.2b*. Then it will be said that g *can be (causally) resolved in (favour of)* T, and T will be called the *resolved event.*

By definition, classical resolution is always in favour of a single resolved event, given initial data about the other events associated with the link. This does not imply anything about the possibility of resolving g in favour of any of these other events. It may be or not be possible to do this.

Suppose the Theorist could in principle resolve T if they were given R, S, U and V, and also resolve S if they were given R, T, U and V. Then the Theorist has to make a choice of resolution and choose one possible resolution and exclude the other. It would not be meaningful classically to resolve g in favour of T and S at the same moment of physiotime, the reason being that these alternative resolutions employ different and inconsistent initial data sets (boundary conditions). Initial data sets are equivalent to information, and it is a self-evident premise that a Theorist can have at most one initial data set from a collection of possible and mutually inconsistent initial data sets at a given moment of physiotime.

There remains one additional exotic possibility. If might be the case that given say R, S and T, both U and V were implied by a knowledge of the link. For example, suppose the link was equivalent to the equation

$$r + s + it + u + iv = 0, \tag{4}$$

where $i = \sqrt{-1}$. Assuming that r, s, t, u and v were always required to be real, then we could always find u and v from a given r, s and t, simply by equating real and imaginary parts. A situation where a given link can be resolved in two or more events given just one initial data set at that link will be called a *fluctuation process.* Fluctuation processes will be excluded from our notion of classical causality. They may have a role in the QM (quantum mechanics) version of causality, which is beyond the scope of this article.

One reason for excluding fluctuation processes is that this guarantees that information flows (in physiotime) from a link into a *single* event. This is related to the concept of *cosmic time* discussed below and to the idea that classical mathematical time has one dimension.
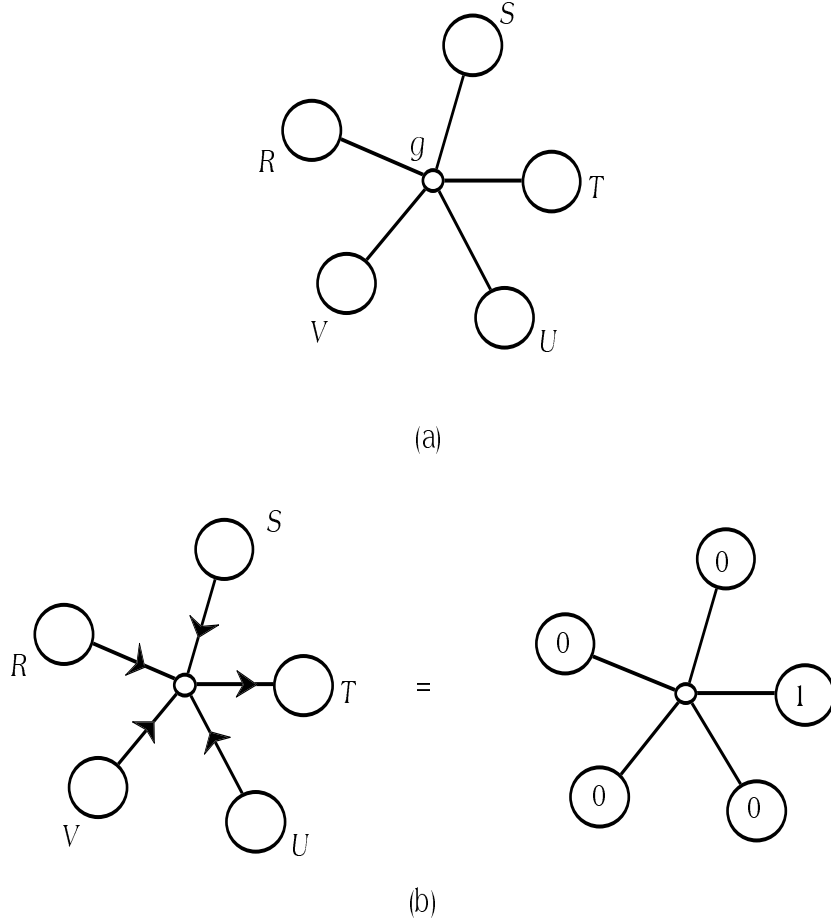
Figure 2: (a) An implicit equation or link involving elements from five events. (b) If the link can be resolved in favour of T then arrows are added as on the LHS. An equivalent diagram is given on the RHS with the ordering of the integers indicating the direction of the resolution.

When theorists discuss models with more than one parameter called a time, all but one of these has to be hidden or eliminated at the end of the day if a classical picture is to emerge.

It may be argued therefore that *the mechanism of classical resolution is the origin of the concept of time, and that time, like causality, is a no more than a convenient theoretical construct designed by the human mind to provide a coherent description of physical reality.* This carries no implication that what we called process time is really a linear time. As we said before, process time is just a convenient label for something which may be quite different to what we believe it to be.

When a choice of resolution exists and is made, then as an additional simplification and provided there is no confusion with other relations to which T may be a party (not shown), the diagram on the LHS of $Fig.2b$ may be replaced by the RHS of $Fig.2b$. Here the numbers zero and one indicate the ordering of the resolution. Because T may be regarded as *caused* by the other events, it can be regarded as later and so has a greater associated discrete time.

Such times will be called *dates*.

In general a link may be a whole collection of relations and equations. If there is just one small part of these equations which does not determine a unique solution fully, i.e., prevents a resolution, then arrows or dates are in principle not permitted. However, under some circumstances it may be reasonable to ignore some part of a dynamical relation in such a way that arrows could be justified as far as the remaining parts were concerned. For example, the microscopic laws of mechanics appear to be resolvable forwards and backwards in time (i.e., are reversible) provided the "small" matter of neutral kaon decays and the thermodynamic arrow of time (which could involve the gravitational field [4]) are ignored.

In $Fig.1a$, D is the *complete cause* of R; in $Fig.2b$, R is a *partial cause* of T. The *complete cause* of T is the collection of events R, S, U and V, but only for this choice of resolution. The Theorist could decide to alter boundary conditions so that T was no longer regarded as the resolved event.

Having outlined our ideas on functions and links, we shall apply them now to discrete spacetime.

5   Discrete spacetime

Fourier's *principle of similitude* states that a system S' similar to but smaller than another system S should behave like S. It is the physicist's analogue of continuity in mathematics, and is of course an erroneous principle when applied to matter, as evidenced by the observation of atoms and molecules. It is generally supposed that this principle will also break down in the microscopic description of space and time. Classical GR (general relativity) may therefore be an approximation, albeit a remarkably good approximation, to some model of space and time which is not intrinsically a four-dimensional pseudo-Riemannian manifold.

There have been numerous suggestions concerning the fundamental nature and meaning of space and time, such as twistor theory, point set theory, etc., [5] and each of these suggestions makes a specific set of mathematical assumptions about spacetime. Likewise, in this article a specific view of space time and dynamics is proposed and its consequences explored. Of course, there is an important question concerning the use of classical or quantum physics here. In this paper the proposals are based on classical ideas and the ramifications of quantum physics are explored elsewhere.

From before the time of Newton, physicists took the view that material objects have definite spatial positions at definite times. In the 20th Century theorists went further and developed to the extreme the Wellsian or block Universe [6] view that space and time exist in some physically meaningful sense, even in the absence of matter. Whatever that sense is, be it a physical one or simply an approximate relationship of sorts between more complex attributes of reality, most physicists agree on the prime status of spacetime as the arena in which or over which physical objects exist. This is certainly the case for classical physicists and to various degrees for quantum physicists.

In this paper the focus is on a classical description of a *discrete* spacetime structure. Discreteness is considered here for several reasons. First, as mentioned above, it would be too much to hope that Fourier's principle of similitude should apply to spacetime and not to matter. Second, there is a strong feeling in the subject of quantum gravity that the Planck scales are significant. Third, discreteness has the advantage over continuity in being less mathematically restrictive. Theories based on discrete principles can usually encompass the

properties of those based on continuous principles via appropriate limit processes, yet retain features which cannot occur in the continuum.

Discrete spacetime structure and its relationship to causality has been discussed by a number of authors, notably by Sorkin et al [7] and Finkelstein et al [8]. A basic difference between those approaches and that taken here is that no *a priori* underlying spacetime manifold is assumed here.

Proposition 1: classical spacetime may be modelled by some discrete set $\mathcal{S}$.

Elements of $\mathcal{S}$ will be denoted by capital letters such as P, Q, R, etc. $\mathcal{S}$ will be called a *spacetime* and its elements referred to as *events* even if subsequently $\mathcal{S}$ turns out to have only an indirect relationship with the usual four-dimensional spacetime of physics. What these events mean physically depends on the model. It is simplest to think of events as labels for mathematical structures representing the deeper physical reality associated with process time.

Events are meaningful only in relationship with each other and it is meaningless to talk about a single event in spacetime without a discussion of how the Theorist relates it to other events in spacetime. This is done by specifying the *links* or relationships between the events. Links are as important as the events themselves and it is the totality of links and events which makes up our spacetime dynamics. This should include all attributes relating to matter and gravitation, and it is in principle not possible to discuss one without the other.

The structure of our spacetime dynamics is really all there is; links and events. No preordained notion of metrical causality involving spacelike and timelike intervals is assumed from the outset. All of that should emerge as part of the implications of the theory. In classical continuous spacetime theories, on the other hand, the metric is usually assumed to exist independently of any matter, even before it is found via the equations of GR. As we said before, this metric carries with it lightcone structure and other pre-ordained attributes of causality.

Discrete spacetime carries with it the astonishing possibility of providing a natural explanation for length, area and volume. According to this idea, attributed to Riemann [7], these are simply *numerical counts* of how many events lie in certain subsets of spacetime. Discreteness may also provide a natural scale for the elimination of the divergences of field theory, and permits all sorts of novelties to occur which are difficult if not impossible to build into a manifold. Recently, the study of spin networks in quantum gravity has revealed that quantization of length, area and volume can occur [9].

6 Event state space

In CM the aim is usually to describe the temporal evolution of chosen dynamical degrees of freedom. These take on many possible forms, such as position coordinates or various fields variables such as scalar, vector and spinor fields. In our approach, we associate with each event P in a given spacetime $\mathcal{S}$ an internal space Ω_P of dynamical degrees of freedom called *event state space*. This space could be whatever the Theorist requires to model the situation. Moreover, it could be different in nature at each event P in this spacetime. Elements of Ω_P will be denoted by lower case Greek letters, such as ξ_p, λ_P etc. and a chosen element ξ_P of Ω_P will be called a *state of the event P*.

Elementary examples of event state spaces are:

i) **Scalar fields:**   a real valued scalar field f on a spacetime $\mathcal{S}$ is simply a rule which assigns at each event $P \in \mathcal{S}$ some real number f_p. This may be readily generalized to complex valued functions. If no other structure is involved then obviously

$$
\begin{aligned}
\Omega_P &= \mathsf{R}_P \quad \forall P \in \mathcal{S}, \\
\xi_P &\equiv f_P \in \mathsf{R}_P,
\end{aligned}
\tag{5}
$$

where R_P is a copy at P of the real line R.

A classical configuration of spacetime in this model would then be some set $\{f_P, \ f_Q, ...\}$. This configuration "exists" at some moment of the Theorist's physiotime but the model itself would not necessarily have any causal ordering. That would have to be determined by the Theorist in the manner discussed below.

ii) **Vector fields:**   suppose at each event P there is a copy V_P of some finite dimensional vector space V with elements $\mathbf{v} \in V$, etc. Then a vector field is simply a rule which assigns at each event P some element $\mathbf{v}_P \in V_P$.

iii) **Group manifolds:**   at each event P we choose a copy G_P of some chosen abstract group, such as Z_2 or $SU(3)$ to be our event space.

iv) **Spin networks:**   A spin network is a graph with edges labelled by representations of a Lie group and vertices labelled by intertwining operators. Spin networks were originally invented by Penrose in an attempt to formulate spacetime physics in terms of combinatorial techniques [10] but they may also defined as graphs embedded in a pre-existing manifold [11]. The state event space at each event P is a copy of $SU(2)$. In this particular model the sort of events we are thinking of in spacetime would be associated with the geometrical links of a triangulation and the links (the dynamical relationship between our events) would be associated with the geometrical vertices of the triangulation, that is, with the intertwining operators.

6.1   Neighbourhoods and local environments

For physically realistic models the number of events in the corresponding spacetime $\mathcal{S}$ will be vast, possibly infinite. Sorkin et al [7] give a figure of the order 10^{139} per cubic centimetre-second, assuming Planck scales for the discrete spacetime structure. We shall find it useful to discuss some examples with a finite number of events for illustrative purposes.

In our spacetime diagrams we will follow the convention established for functions in the previous section; large circles denote events and small circles denote links, with lines connecting events and links. Before any temporal resolution is attempted, no arrows can be drawn. $Fig.3a$ shows a finite spacetime with 14 events and 9 links.

Definition 2:   The (*local*) *environment* $\mathcal{E}_A$ *of an event* A is the subset of links which involves A, that is, all those links to which A is party, and the *degree of an event* is the number of elements in its local environment.

For example, from $Fig.3a$, the local environment of the event labelled P is the set of links $\mathcal{E}_P \equiv \{f, g, h\}$, and so P is a third degree event.

Definition 3: The *neighbourhood $\mathcal{N}_A$ of an event A* is the set of events linked to A via its environment.

For example, from $Fig.3a$ the neighbourhood of event P is the set of events $\mathcal{N}_P \equiv \{Q, R, S, T, U, V\}$, and Q, R, S, T, U and V are the *neighbours* of P.

Definition 4: The *domain $\mathcal{D}_f$ of a link f* is the set of events involved in that link, and the *order of a link* is the number of elements in its domain.

For example, from $Fig.3a$, $\mathcal{D}_f \equiv \{Q, R, P, U\}$ and so f is a fourth order link.

The local environment of an event will be determined by the underlying dynamics of the spacetime, i.e. the assumed fundamental laws of physics. Currently these laws are still being formulated and discussed, so only a more general (and hence vague) discussion can be given here with some simple examples. A spacetime and its associated structure of neighbourhoods and local environments will be called a *spacetime dynamics.*

7 Kinds of spacetime dynamics

There are two interrelated aspects of any spacetime dynamics: i) the nature of the event state space associated with each event and ii) the nature of the links connecting these events. The way in which events are related to each other structurally via the links will be called the *local discrete topology.* If this discrete topology has a regularity holding for all links and events, such as that of some regular lattice network, then we shall call this a *homogeneous* discrete topology. Otherwise it will be called *inhomogeneous.*

We envisage three classes of classical spacetime dynamics:

1. **Type A:** spacetime dynamics with a homogeneous and fixed discrete topology with a variable[2] event state configuration which does not affect the discrete topology. This corresponds to (say) field theory over Minkowski spacetime;

2. **Type B:** spacetime dynamics with inhomogeneous but fixed discrete topology with a variable event state configuration which does not affect the discrete topology. This corresponds to field theory over a fixed curved background spacetime, such as in the vicinity of a black hole;

3. **Type C:** spacetime dynamics with discrete topology determined by the event state configuration; this corresponds to GR with matter.

Type A and B spacetime dynamics are relatively easy to discuss. Once a fixed discrete topology is given this provides the template or matrix for the Theorist to "slot in" the causal patterns associated with initial event state configurations (initial data sets). In this sense, types A and B are not genuinely background free, but they are independent in a sense of any preordained Lorentzian metric structure.

Type C presents an altogether more interesting scenario to discuss and demonstrates the basic issue in classical GR which is that the spacetime dynamics should determine its own structure, including its topology.

GR without matter of any sort does not make sense in our approach, because we need to specify event state spaces in order to define the links. Spins are needed to specify spin

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[2]i.e., variable in physiotime.

(a)

(b)

Figure 3: (a) $\{f, g, h\}$ is the local environment of P and $\{Q, R, S, T, U, V\}$ is its neighbourhood. $\{Q, P, R, U\}$ is the domain of f. (b) A chosen initial data set $\{O, P, Q, U\}$ is shaded grey. Its full implication (or future) is $\{R, S, T, V, W, X, Y, Z\}$. Its (absolute) past $\{M, N\}$ is inaccessible from this initial data set and cannot be influenced by any changes on the initial data set.

networks, for example. In our approach, gravitation is intimately bound up with discrete spacetime topology.

A spacetime diagram need not be planar. Indeed, there need not be any concept of spacetime dimension at this stage. Sorkin et al [7] suggest that at different scales, a given discrete spacetime structure might appear to be approximated by different continuous spacetime

dimensions, such as 26, 10, or 4, depending on the scale.

8   Classical resolution and causal structure

The PPM model was introduced as an approach to the modelling of process time. Working in physiotime, the Theorist first decides on a spacetime dynamics and from an initial data set then determines a consistent event state at each event in the spacetime. Because of the existence of the links, however, these event states cannot all be independent, and this interdependence induces our notion of causality, as we now explain.

First, restrict the discussion to Type A and B spacetimes and suppose that each link is *fully resolvable*. By this is meant that if the order of a link is n and the event states are specified for any $n-1$ of the events in the domain of the link, then the remaining event space in the domain is uniquely resolved.

Example 1:   An example of a fully resolvable link is the following: Let f be a link of order r, with domain $\mathcal{D}_f \equiv \{P_1, P_2, ..., P_r\}$. Suppose the event space at P_i is a copy G_i of some group G such as Z_2 or $SU(n)$ and suppose the link is defined by

$$f : g_1 g_2 ... g_r = e, \tag{6}$$

where $g_i \in G_i$ and e is the group identity[3]. Then clearly,

$$g_1 = g_r^{-1} g_{r-1}^{-1} ... g_2^{-1} \tag{7}$$

and similarly for any of the other events states. In other words, we can always resolve any one of the events in $\mathcal{D}_f$ uniquely in terms of the others.

Now suppose the Theorist chooses one link, such as f in $Fig.3a$, which is a fourth order link. Then its domain $\mathcal{D}_f$ can be identified immediately from the spacetime dynamics to be $\{P, Q, R, U\}$. The Theorist is free to specify the states at three of these without any constraints, and this represents an initial data set called $\mathcal{S}_0$. Suppose these are events P, Q and U.

If now the structure of the link f is such that there is only one possible solution in $\Omega(R)$, then we have a classical resolution of alternatives[4] and the emergence of a causal structure. We can use the language of dynamics and say that events P, Q and U *cause* R and denote it by a diagram such as $Fig.2b$. But it should be kept in mind that the Theorist has *decided on which three sets to use as an initial data set.* Our interpretation of causality is that it is dictated partly by the spacetime dynamics and partly by the choices made by the Theorist. In general, classical resolution involves using information about $n-1$ event states at an nth order link to determine the state of the remaining event in the domain of that link.

An initial data set may involve more than one link, such as shown in $Fig.3b$. In that diagram, events are labelled by integers representing the discrete times at which their states may be fixed. Events in the chosen initial data set are labelled with a time 0 and shaded grey. These are the events $\{O, P, Q, U\}$.

Given an initial data set $\mathcal{S}_0$, the Theorist can then use the links to deduce the *first (or primary) implication* $\mathcal{S}_1$. This is the set labelled by integer time 1 in $Fig.3b$, and consists of

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[3]We may use a matrix representation to define products of elements from different copies of G.

[4]i.e., the alternatives which form the event space $\Omega(R)$.

events $\{R, V\}$. The event state at each of the events in the primary implication is determined from a knowledge of the event states on the initial data set, assuming the links do indeed permit a classical TRA. It could be the case that the primary implication is the empty set.

Given a knowledge of the event states on $\mathcal{S}_0$ and its primary implication $\mathcal{S}_1$, then the *second* (or *secondary*) *implication* $\mathcal{S}_2 \equiv \{W, T\}$ can now be found and its events labelled by the discrete time 2. This process is then repeated until the *full implication* $\mathcal{S}_\infty$ of $\mathcal{S}_0$ is determined. In $Fig.3b$ this is the set $\{R, S, T, V, W, X, Y, Z\}$.

In this example it is assumed that each of the links is fully resolvable. If any link is not fully resolvable, it may still be possible to construct a non-empty full implication for certain initial data sets.

Several important concepts can be discussed with this example.

8.1   The future of an initial data set

The full implication of an initial data set consists of those events whose event states follow *from* given initial conditions, and so it is not unreasonable to call the full implication of any initial data set the *future* of that set. Whilst the term *"from"* in the preceding sentence refers to a process of inference or implication carried out in physiotime by the Theorist, the result is that diagram $Fig.3b$ for example now carries *dates* (or equivalently *arrows*) as a consequence. This ordering may now be regarded as an attribute of the mathematical model rather than of physiotime, and this is the *mathematical arrow of time* referred to previously. The resulting structure can then be used to represent phenomena in process time.

However, some caution should be taken here. We may encounter spacetime dynamics which are equivalent to *reversible dynamics* in continuous time mechanics. The full implication of some initial data sets for such spacetime mechanics may propagate both into what we would normally think of as the conventional future *and* into the conventional past. This is in accordance with the general problem encountered with any reversible dynamics: we can never be sure which direction is the real future and which is the real past unless external information is supplied to tell us.

An example of a reversible discrete topology is given in $Fig.4a$. This is the topology of the discrete time harmonic oscillator [12]. Assuming the links are fully resolvable, we see that the initial data set shown (shaded and labelled 0) has a full implication which extends to the right and to the left of the diagram $Fig.4b$, i.e. into the conventional past and future.

8.2   Inaccessible events

In $Fig.3b$ the set $\{M, N\}$ is inaccessible from the given initial data set $\{O, P, Q, U\}$, that is, its intersection with both the initial data set and its full implication is the empty set. Such an inaccessible set may be interpreted in a number of ways. It could be thought of as the *absolute past* of the initial data set $\{O, P, Q, U\}$ because in this particular example, the initial data set implies nothing about $\{M, N\}$, but specifying the states at M and N would fix Q. In other examples such inaccessible events could be interpreted as beyond an event horizon of some sort. The general feature of inaccessible events is that they cannot be affected by any changes to the event states in an initial data set.

Archaeology is our term for the process of reconstructing portions of an absolute past from new and limited information added to some initial data set. It means the process whereby
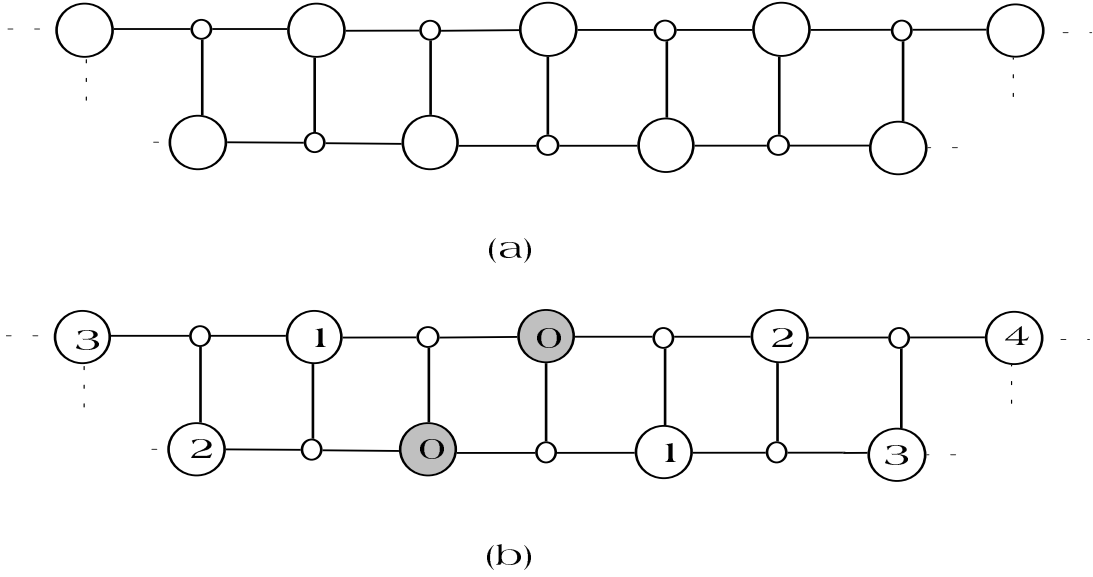
Figure 4: An example of a reversible spacetime dynamics, the discrete time harmonic oscillator. (a) The basic discrete topology. (b) The chosen initial data set (shaded) needs two real numbers, equivalent to initial position and velocity. The full implication runs into the left (conventional past) and to the right (conventional future), and there are no inaccessible events.

specifying one or more event states in an absolute past has the immediate consequence that the Theorist can deduce even more information about that past.

Example 2:   Consider a spacetime dynamics based on the infinite regular triangular topology shown in $Fig.5$, with an initial data set shaded and labelled by 0. Its first, second and third implications are labelled by $1, 2$ and 3 respectively. The full implication of the initial data set in fact extends to infinity on the right. Suppose the Theorist now determines in some way or decides on the state at the event shaded and dated -1 in $Fig.5$. Then the events in the past of the initial data set dated by -1 and unshaded can now be resolved, thereby extending the original full implication one layer to the left of the initial data set. Just one extra piece of information can trigger an implication with an infinite number of elements. This process of retrodiction is called *archaeology* for obvious reasons[5], and for this spacetime could be continued indefinitely into the past. The Theorist can, by providing new initial data in this way, eventually cover the entire spacetime as the union of an extended initial data set and its full implication.

9   Causal Propagators and the speed of causality

Suppose we have been given an initial data set $\mathcal{S}_0$ and have worked out its full implication $\mathcal{S}_\infty$. Now pick any event P in $\mathcal{S}_0$ and change its state ψ_P. The consequence of this is to

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[5]The act of finding just a few Roman artefacts in a field may lead an Archaeologist to the conclusion that there had been a Roman settlement there.

Figure 5: The initial data set is shaded and labelled by zeros. Its first, second and third implications are labelled by 1,2, and 3 respectively. The past of this initial data set is the set of all events to the left of the zeros. By providing just one extra data event (shaded and labelled -1) the full implication can be extended to include the events unshaded and labelled by -1. Similarly, by providing another extra data set (shaded and labelled by -2), more evidence about what the past must have been can be deduced.

change the states in some events in $\mathcal{S}_\infty$, but not necessarily in all events in $\mathcal{S}_\infty$. The subset $\mathcal{P}_P(\mathcal{S}_0)$ of $\mathcal{S}$ consisting of all those events changed by the change in P will be called the *causal propagator* associated with P and $\mathcal{S}_0$. P will be called the *vertex of the propagator.*

   We note the following:

1. for any event Q in an initial data set $\mathcal{S}_0$, the causal propagator $\mathcal{P}_Q(\mathcal{S}_0)$ lies entirely within the full implication $\mathcal{S}_\infty$ of $\mathcal{S}_0$;

2. A causal propagator depends on a vertex *and* on an associated initial data set;

3. A causal propagator divides spacetime into three sets: the vertex, those events which cannot be affected by any change at the vertex, and those events which could be changed. This structure is rather like the lightcone structure in special relativity which separates events into those which are timelike, lightlike, or spacelike relative to the vertex of the lightcone. In our context, we could in some sense talk about the *speed of causality*, analogous to the speed of light in relativity, as the limiting speed with which disturbances could propagate over our spacetime.

   Example 3 :   As an example we give a spacetime lattice of Type A labelled by two integers m, n running from $-\infty$ to $+\infty$. The state space Ω_n^m at each event P_n^m is the set
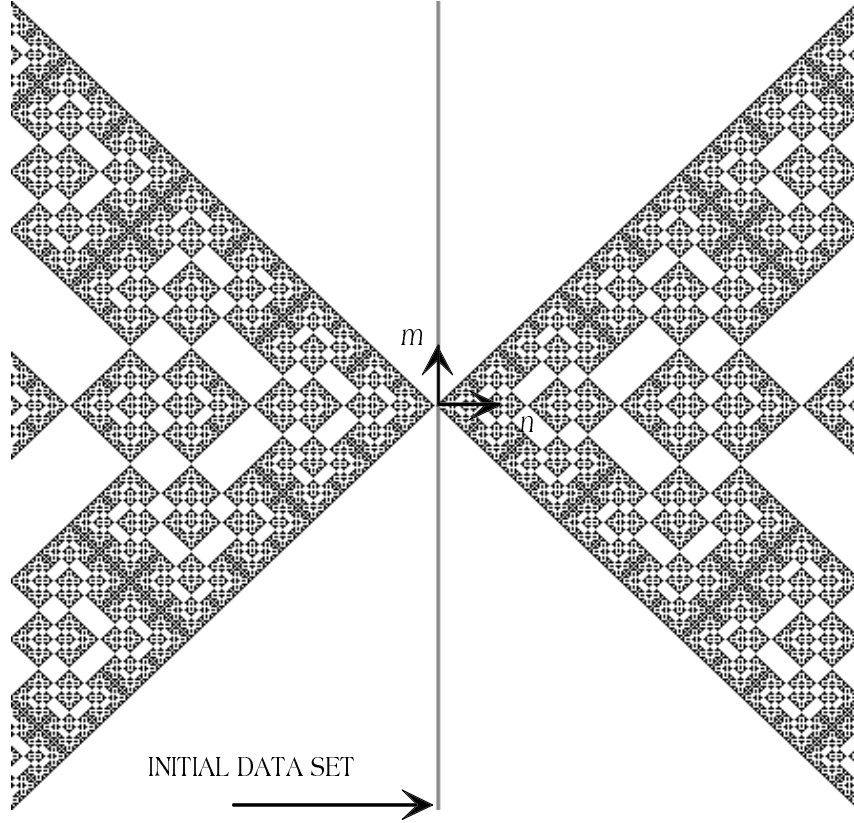
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Figure 6: The causal propagator associated with the reversible spacetime dynamics and initial data set in Example 3, with vertex at $(0,0)$. In this diagram time appears superficially to run left to right or right to left, but actually the causal flow is from the initial data set outwards to the left and to the right.

$\{+1, -1\}$ and a state at P_n^m will be denoted by ψ_n^m. The links are given by the equations

$$\psi_n^m \psi_n^{m+1} \psi_n^{m-1} \psi_{n-1}^m \psi_{n+1}^m = 1, \quad -\infty < m, n < \infty.$$

Now choose an initial data set

$$\psi_0^m = \psi_1^m = +1, \quad -\infty < m < \infty.$$

This corresponds to selecting the index m as a spatial coordinate and the index n as a timelike coordinate. The initial data set is then equivalent to specifying the initial values and initial time derivatives of a scalar field on a hyperplane of simultaneity in a two dimensional spacetime.

Because this spacetime dynamics is reversible, the full implication of this initial data set is the entire spacetime minus the initial data set.

Now consider changing the state at $m = n = 0$ from $\psi_0^0 = +1$ to $\psi_0^0 = -1$. In $Fig.6$ we show a bitmap plot of all those events whose state is changed by the change in the event $(0,0)$. The structure looks just like a lightcone, with complex fractal-like patterns developing inside the retarded and advanced parts of the lightcone. The speed of causality in this example is evidently unity if we interpret the indices in the manner discussed above.
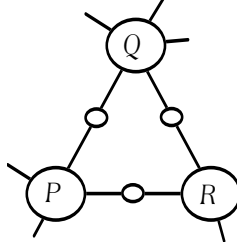
Figure 7: If any spacetime dynamics has a causal loop such as PQR embedded in it then no global cosmic time ordering is possible.

10   Cosmic time

For some spacetime dynamics a global temporal ordering can constructed by assigning an integer to each event as follows. If P is earlier then Q (i.e. P is a partial or complete cause of Q) then some integer p is assigned to P and some integer q to Q such that $p < q$. These integers are called *dates* above.

   If it is possible to find a consistent ordering over the whole of $\mathcal{S}$ based on the above rule then we may say that a *cosmic time* exists for that spacetime. A cosmic time cannot be constructed for a finite spacetime dynamics if there are no events of degree 1. There are two situations where a cosmic time may be possible: either the spacetime is finite with one or more events of degree one, or the spacetime is infinite. However, these are not sufficient properties to guarantee a cosmic time can be constructed.

   It is possible to find spacetime dynamics for which more than one cosmic time pattern can be established.

10.1   Causal (timelike) loops:

Fig.7 shows part of a spacetime dynamics containing a closed *causal (timelike) loop.* No cosmic time can be found for such a spacetime. This corresponds to the situation in GR where the existence of a closed timelike loop in a spacetime precludes the possibility of finding a global cosmic time coordinate for that spacetime.

10.2   Spacelike hypersurfaces

In the spacetime depicted in *Fig*.3 the concept of metric was not introduced. Nevertheless, it is possible to give a definition of a *spacelike (hyper)* surface in this and other spacetime. Whether this corresponds to anything useful depends on the details of the spacetime dynamics.

   Definition 5:   A *spacelike hypersurface* $\sum$ of a spacetime $\mathcal{S}$ is any subset of $\mathcal{S}$ which would have a consistent full implication if it were used as an initial data set.

   We have already used the term *initial data set* for such subsets. However, not all initial data sets need be consistent. A spacelike hypersurface is just a consistent initial data set.

   There may be many spacelike hypersurfaces associated with a given spacetime and they need not all be disjoint. The definition of spacelike hypersurface involves a choice of causation

by the Theorist. In general there may be more than one spacelike hypersurface passing through a given event, and it is the Theorist's choice which one to use. The possible non-uniqueness of spacelike hypersurface associated with a given event is desirable, because this is precisely what occurs in Minkowski spacetime, where there is an infinite number of spacelike hypersurfaces passing through any given event, corresponding to hyperplanes of simultaneity in different inertial frames.

11   Causal sets

Causal sets are sets with some concept of ordering relationship, which makes them suitable for discussions concerning causality [7]. This presupposes some pre-existing temporal structure independent of any dynamical input. This is not a feature of our dynamics, where causal structure emerges only after the Theorist has chosen the initial data set.

12   Concluding remarks and summary

The PPM model leaves certain fundamental questions unanswered, such as the origin of physiotime and whether process time is a meaningful concept. However, once these are accepted as given and the model used in the right way, then causality and time itself emerge as observer (Theorist) oriented concepts. Many if not all of the phenomena associated with metric theories of spacetime can be recovered, which suggests that further investigation into this approach to spacetime dynamics may prove fruitful.

A good question to ask is: where would Lorentzian causality come from in our approach? The answer is that it is embedded or encoded in the definition of the links. Only when full implications are worked out from given initial data sets would it be noticed that the dynamics itself naturally forces certain patterns to emerge and not others. Only at that stage would the Theorist would recognize an underlying bias in the dynamics in favour of certain more familiar interpretations.

For example, given the continuous time equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - m^2\right)\varphi\left(t, x\right) = 0, \tag{8}$$

the notation suggests that t is a time, so we could attempt to solve it with initial data on the hyperplane $t = 0$. However, it would soon emerge that evolution in t gave runaway solutions. In other words, the equation itself would carry the information that it would be wiser to define initial data on the hyperplane $x = 0$. We would not need to invoke the spurious concept of Lorentzian signature metric embedded in spacetime to discover this. At this stage we might suddenly realise that we had by some chance interchanged the symbols for time and space in an otherwise ordinary Klein-Gordon equation with a real mass. So rather than deal with (8), which behaves like a Klein-Gordon equation with an imaginary mass, we would simply interchange the symbols t and x and then define initial data on the hyperplane now defined by $t = 0$.

13　Acknowledgment

References

[1] G. J. Whitrow, *The Natural Philosophy of Time*, Clarendon Press, Oxford, 2nd Edition (1980)

[2] A.G. Howson, *A Handbook of Terms Used in Algebra and Analysis*, CUP (1972)

[3] Gerard 't Hooft, *Quantum Gravity as a Dissipative Deterministic System*, gr-qc/9903084

[4] Bernard S. Kay, *Entropy defined, entropy increase and decoherence understood, and some black-hole puzzles solved,* hep-th/9802172

[5] J. Butterfield and C. J. Isham, *Spacetime and the Philosophical Challenge of Quantum Gravity,* gr-qc/9903072

[6] Huw Price, *Time's Arrow*, Oxford University Press (1997)

[7] D. Meyer L. Bombelli, J. Lee and R. Sorkin, *Space-Time as a Causal Set*, Phys. Rev. Lett., 59(5), pp521–524 (1987); D. P. Rideout and R.D. Sorkin, *A Classical Sequential Growth Dynamics for Causal Sets,* gr-qc/9904062

[8] Graham Frye David Finkelstein and Leonard Susskind, *Space-time code. V*, Physical Review D, 9(8), pp2231–2236 (1974)

[9] Carlo Rovelli, *Strings, loops and others: a critical survey of the present approaches to quantum gravity*, gr-qc/9803024

[10] Carlo Rovelli and Lee Smolin, *Spin networks and quantum gravity*, gr-qc/9505006

[11] John C. Baez, *Spin networks in nonperturbative quantum gravity,* gr-qc/9504036

[12] G. Jaroszkiewicz and K. Norton, *Principles of discrete time mechanics: I. Particle systems*, J. Phys. A: Math. Gen., 9(7), pp 3115-3144 (1997)